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Asymptotic analysis of Gaussian focused ultrasonic beams of circular symmetry

Theodore E Matikas

National Research Council, Materials Directorate, Wright Laboratory, Wright-Patterson Air Force Base, OH 45433-7817, USA

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Abstract. In this paper, a model of a focused beam with circular symmetry is presented. The acoustic field is defined by a Gaussian surface velocity distribution along the emitter immersed in a fluid. The pressure field is described by a Fourier integral representation, and is evaluated asymptotically using a generalized steepest descent procedure. Simple analytical expressions for the acoustic pressure along the axis are obtained, and the variation of the pressure field as a function of distance from the emitter is illustrated.

1. Introduction

Theoretical studies consecrated to evaluation of the ultrasonic field generated from focused transducers were developed in the past. With the assumption of small wavelengths with respect to the width of the emitter (the hypothesis of short waves or high frequencies), those beams described mathematically by the Rayleigh integral or by a Fourier representation were evaluated using the Fresnel approximation [1–6], the parabolic approximation [7], or the paraxial approximation [8, 9].

O'Neil [1] modelled a focused acoustic beam by considering a concave transducer vibrating at a uniform normal velocity. Under the hypothesis of short waves, he approximated the potential of velocities using Green's formulation, which is only valid in the case of a plane acoustic source. This model finally gives an analytical expression for both acoustic pressure and intensity along the axis of symmetry of the beam and in the focal plane.

Cavanagh and Cook [2, 3] modelled a focused acoustic beam by using a plane transducer together with a lens positioned at a certain distance from the transducer. In order to obtain an analytical solution for the acoustic pressure field, the authors used the Rayleigh–Sommerfeld formulation based on Huygen's principle, which consists of describing the acoustic pressure outside the source as a superposition of divergent spherical waves. The obtained analytical expression for the pressure field was in an integral form and has been evaluated using Fresnel's approximation and by means of Laguerre–Gaussian functions. Finally, the acoustic pressure was obtained using a simple numerical method.

Lucas and Muir [5] considered a concave transducer and used the Rayleigh–Sommerfeld formulation. Using Fresnel's approximation and by means of the hypothesis of high frequencies, they obtained a well-collimated beam. The authors finally evaluated the acoustic pressure on the acoustic axis.

Filipczynski and Etienne [6] considered a spherical transducer vibrating with a Gaussian velocity profile. By means of the hypothesis of short waves and Fresnel's approximation, the Rayleigh integral has been evaluated and the pressure of the focused Gaussian beam on the acoustic axis was finally obtained.

Pott and Harris [8, 9] modelled a focused Gaussian beam by considering an acoustic monopole with complex spatial coordinates. Using the paraxial approximation, the normal velocity of the monopole was made to correspond to a Gaussian distribution. The paraxial approximation consists of taking into account a restricted area of the beam; in that area only, the beam appears to be of Gaussian type. This method of modelling a focused beam does not imply any assumption regarding the very structure of the beam, but is based on spatial limitation of the beam. Contrary to the paraxial approximation, in the cases of the parabolic approximation [7] or Fresnel's approximation [1–6], the beam is of Gaussian type by the definition of its structure. Both paraxial and parabolic approximations and also Fresnel's approximation are equivalent in terms of obtaining the final expression for the acoustic pressure.

More recently, a two-dimensional model has been proposed for studying Gaussian focused beams. By using the steepest descent procedure, analytical expressions for the pressure field were obtained for any point in space, including the neighbourhood of the caustic and the focal point [10–13].

In order to complete our previous studies, we present here a model of a Gaussian focused beam with revolution

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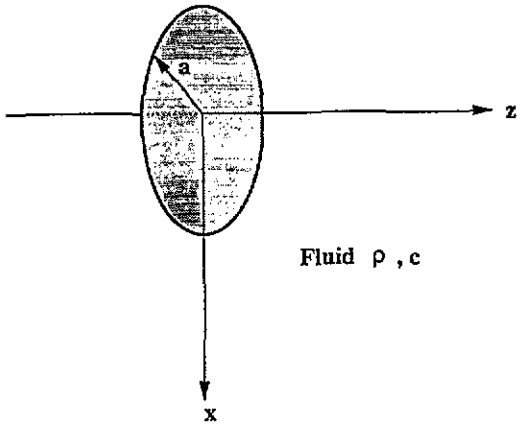

Figure 1. Configuration of the disposition.

symmetry. This paper is not based on any of the above assumptions (Fresnel, paraxial or parabolic), which consist of considering a highly directional ultrasonic beam. The incident beam is defined here by its normal velocity distribution along the plane-emitter placed in a fluid medium. The Fourier integral representation of the acoustic field along the axis is evaluated asymptotically, by means of a generalized method of steepest descent.

2. The model

Let us consider a Gaussian focused beam generated by a plane, circularly symmetric emitter placed in a fluid of mass density ρ and sound velocity c (figure 1).

In order to describe the circular symmetry, we use polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$k_x = k_r\cos\varphi \qquad k_y = k_r\sin\varphi.$$

The Gaussian normal velocity distribution of particles along the emitter plane is given by

$$V_n(r,0) = V_0 e^{-(r/a)^2} e^{-ik\sin\theta_0(r^2/a)} e^{-i\omega t} \tag{1}$$

where a is the radius of the emitter, θ_0 the focalization angle, $k = \omega/c$ is the wavenumber in the fluid, ω is the angular frequency, and $r = (x^2+z^2)^{1/2}$ is the spatial variable.

The acoustic pressure in a plane situated at a distance z from the emitter (the x–y plane), is given by plane-wave superposition in the form of a Fourier integral:

$$P(r,z) = \int_0^\infty k_r A(k_r) J_0(rk_r) e^{ik_z z}\, dk_r \tag{2}$$

with $k_z = (k^2 - k_r^2)^{1/2}$ where the function k_z is chosen real and positive for $|k_r| < k$ and imaginary and positive for $|k_r| > k$, and $A(k_r)$ is the Fourier transform of $P(r,0)$:

$$A(k_r) = \frac{1}{2\pi}\int_0^\infty rP(r,0)J_0(rk_r)\, dr \tag{3}$$

with $J_0(rk_r)$ the Bessel function of order zero, defined by

$$J_0(rk_r) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-irk_r\cos(\theta-\varphi)]\, d\theta. \tag{4}$$

From the equation $\operatorname{grad} P = -\rho\partial V_n/\partial t$, we obtain $P(r,0) = V_n(r,0)\rho\omega/k_z$ and thus

$$A(k_r) = \frac{\rho\omega V_0}{2\pi k_z}\int_0^\infty re^{-\beta^2 r^2} J_0(rk_r)\, dr \tag{5}$$

where

$$\beta^2 = \frac{1}{a^2} + \frac{ik\sin\theta_0}{a}.$$

By expanding the Bessel function as a power series, equation (5) becomes

$$A(k_r) = \frac{\rho\omega V_0}{2\pi k_z}\int_0^\infty re^{-\beta^2 r^2}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{rk_r}{2}\right)^{2k}}{k!\Gamma(k+1)}\, dr$$
$$= \frac{\rho\omega V_0}{2\pi k_z}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1)}\frac{(k_r)^{2k}}{2^{2k}}\int_0^\infty e^{-\beta^2 r^2} r^{2k+1}\, dr. \tag{6}$$

On changing the variables such that $t \to \beta^2 r^2$, it follows that

$$A(k_r) = \frac{\rho\omega V_0}{2\pi k_z}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1)}\left(\frac{k_r}{2}\right)^{2k} \times \frac{1}{2\beta^{2k+2}}\int_0^\infty e^{-t}t^k\, dt. \tag{7}$$

From the definition of the gamma function

$$\Gamma(k) = \int_0^\infty e^{-t}t^{k-1}\, dt$$

equation (7) can be simplified to

$$A(k_r) = \frac{\rho\omega V_0}{4\pi k_z\beta^2}\sum_{k=0}^{\infty} -\left[\left(\frac{k_r^2}{4\beta^2}\right)^k \Big/ k!\right]. \tag{8}$$

By using the identity

$$\sum_{k=0}^{\infty}\frac{X^k}{k!} = e^x$$

we obtain

$$A(k_r) = \frac{\rho\omega V_0}{4\pi k_z}\frac{1}{a^{-2} + i(k/a)\sin\theta_0} \times \exp[-k_r^2/4(a^{-2} + ika\sin\theta_0)]. \tag{9}$$

Thus, the acoustic pressure field (equation (2)) is given by the expression

$$P(\bar r,\bar z) = \frac{\rho c V_0 (ka)^2}{4\pi(1+ika\sin\theta_0)} \times \int_0^\infty \frac{\bar k_r J_0(ka\bar r\bar k_r) e^{-\bar k_r^2(ka)^2/4(1+ika\sin\theta_0)} e^{i(ka)\bar k_z\bar z}}{(1-\bar k_r^2)^{1/2}}\, d\bar k_r \tag{10}$$

where $\bar{r} = r/a$, $\bar{z} = z/a$, $\bar{k}_r = k_r/k$ and $\bar{k}_z = k_z/k$ are non-dimensional quantities.

Finally, by assuming the hypothesis of short waves ($ka \gg 1$), the pressure field is written in the form

$$P(\bar{r},\bar{z}) = \frac{\rho c V_0 ka}{\mathrm{i}4\pi \sin\theta_0} \times \int_0^\infty \left\{ \bar{k}_r J_0(ka\bar{r}\bar{k}_r)\exp[-\bar{k}_r^2/4\sin^2\theta_0] \times \exp\left[\mathrm{i}ka\left((1-\bar{k}_r^2)^{1/2}\bar{z} + \frac{\bar{k}_r^2}{4\sin\theta_0}\right)\right]\right\} \times (1-\bar{k}_r^2)^{-1/2}\,\mathrm{d}\bar{k}_r + O(1). \tag{11}$$

Thus, the acoustic pressure on the axis is given by the expression

$$P(0,\bar{z}) = \frac{\rho c V_0 ka}{\mathrm{i}4\pi\sin\theta_0} I(0,\bar{z}) \tag{12}$$

where

$$I(0,\bar{z}) = \int_0^\infty g(\bar{k}_r)\mathrm{e}^{\mathrm{i}kaf(\bar{k}_r)}\,\mathrm{d}\bar{k}_r \tag{13}$$

with

$$g(\bar{k}_r) = \frac{\bar{k}_r \mathrm{e}^{-(\bar{k}_r^2/4\sin^2\theta_0)}}{(1-\bar{k}_r^2)^{1/2}}$$

$$f(\bar{k}_r) = (1-\bar{k}_r^2)^{1/2}\bar{z} + \frac{\bar{k}_r^2}{4\sin\theta_0}.$$

Let us suppose that the characteristic width of the beam, a, is large relative to the emission wavelength, $\lambda = 2\pi/k$, the parameter ka is thus much larger than 1; hence the integral (13) can be evaluated by the asymptotic method of steepest descent. This method consists of replacing the initial integration path by a new path passing through the saddle-points of the phase function of the integral, $f(k_r)$, defined by the equation

$$f'(\gamma_i) = 0$$

or

$$\frac{\gamma_i}{2\sin\theta_0} - \frac{\gamma_i\bar{z}}{(1-\gamma_i^2)^{1/2}} = 0$$

which has three solutions:

$$\gamma_1 = +[1-(2\sin\theta_0\bar{z})^2]^{1/2}$$

$$\gamma_2 = 0$$

$$\gamma_3 = -[1-(2\sin\theta_0\bar{z})^2]^{1/2}.$$

Since the integration path is $[0,+\infty[$, the negative solution is not taken into account.

We distinguish the following two cases.

(i) If $z > a/2\sin\theta_0$ there are real saddle-points at γ_2.

(ii) If $z \leqslant a/2\sin\theta_0$ there are two real saddle-points at γ_1 and γ_2 ($z = a/2\sin\theta_0$ being the position of the focal point).

When a saddle-point is situated near the integration limit, the classic steepest descent method (used in [10–13]) is not applicable. A generalized method [14] is thus applied (see the appendix), in order to evaluate analytically the integral (13).

Thus, from equation (A14) we obtain the contribution of the saddle-point $\gamma_2 = 0$:

$$I = \frac{\mathrm{e}^{\mu\pi\mathrm{i}/2}}{ka|\frac{1}{2\sin\theta_0} - \bar{z}|}\mathrm{e}^{\mathrm{i}ka\bar{z}} \tag{14}$$

with

$$\mu = \operatorname{sgn}\left(\frac{1}{2\sin\theta_0} - \bar{z}\right)$$

and from equation (A13), the contribution of the saddle-point γ_1 is

$$I = \mathrm{e}^{\mathrm{i}kaf(0)+\mathrm{i}\mu\pi/4} \times \left(\frac{\delta_0 - \eta\delta_1\mathrm{e}^{-\mathrm{i}\mu\pi/4}}{(ka)^{1/2}} W_0[(ka)^{1/2}\eta\mathrm{e}^{-\mathrm{i}\mu\pi/4}] + \frac{\delta_1}{ka}\right) \tag{15}$$

where

$$\delta_0 = 0 \qquad \delta_1 = -\frac{\mathrm{e}^{\mathrm{i}\mu\pi/4}g(\gamma_1)}{\eta(f''(\gamma_1))^{1/2}}$$

$$\eta = -(2|f(\gamma_1) - \bar{z}|)^{1/2}$$

and $\mu = 1$ (as $\bar{z} < 1/(2\sin\theta_0)$, this is necessarily so).

In case (i), only the contribution of the saddle-point γ_2 intervenes. Thus, the acoustic pressure for that part of the axis is

$$\frac{P(0,\bar{z})}{\rho c V_0} = \frac{\mathrm{e}^{\mathrm{i}ka\bar{z}}}{4\pi\sin\theta_0\left(\frac{1}{2\sin\theta_0} - \bar{z}\right)}. \tag{16}$$

In case (ii), both saddle-points γ_1 and γ_2 intervene. Thus, the acoustic pressure for that part of the axis is

$$\frac{P(0,\bar{z})}{\rho c V_0} = \frac{-\mathrm{e}^{\mathrm{i}ka\bar{z}}g(\gamma_1)(ka)^{1/2}}{4\pi\sin\theta_0(2f''(\gamma_1)|f(\gamma_1)-\bar{z}|)^{1/2}} \times \left(\frac{1}{(ka)^{1/2}} + (2|f(\gamma_1)-\bar{z}|)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}W_0[(ka)^{1/2}\eta\mathrm{e}^{\mathrm{i}\pi/4}]\right) + \frac{\mathrm{e}^{\mathrm{i}ka\bar{z}}}{4\pi\sin\theta_0\left(\frac{1}{2\sin\theta_0} - \bar{z}\right)} + O(1/ka) \tag{17}$$

with

$$W_0[(ka)^{1/2}\eta\mathrm{e}^{\mathrm{i}\pi/4}] = \exp\left(\frac{\mathrm{i}\eta^2 ka}{2}\right)\mathrm{e}^{\mathrm{i}\pi/4}\int_{(ka)^{1/2}\eta}^\infty \exp\left(\frac{-\mathrm{i}t^2}{2}\right)\mathrm{d}t.$$

Figure 2 illustrates the pressure field on the acoustic axis as a function of distance from the emitter. The maximum value corresponds to the position of the focal point.

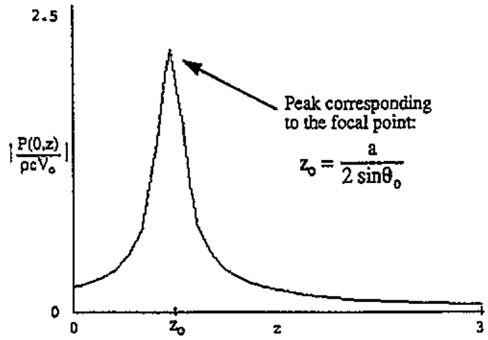

Figure 2. Profile of the acoustic pressure along the axis.

3. Conclusion

In this paper, a model of a Gaussian focused beam generated from an emitter of circular symmetry is presented. The symmetry of revolution is introduced using polar coordinates. The acoustic pressure field is obtained by plane-wave superposition in the form of a Fourier integral. By adopting the hypothesis of short waves, this integral is expanding asymptotically using a generalized steepest descent procedure. Finally, simple analytical expressions for the pressure field along the axis are obtained, and the acoustic pressure as a function of the distance from the plane emitter is illustrated.

Pulsed acoustic fields can be used to validate the model (which considers the monochromatic regime). The technique consists of Fourier transforming the experimental signals and, by selecting the appropriate frequency, experimentally evaluating the acoustic pressure field. This work is the objective of a companion paper.

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Appendix. Asymptotic analysis of integrals in the case of a saddle-point situated near zero; zero being one of the integration bounds

Let us consider the integral

$$I(K) = \int_0^\infty g(t)\mathrm{e}^{\mathrm{i}Kf(t)}\,\mathrm{d}t \qquad \text{(A1)}$$

with $K \gg 1$. γ is the saddle-point of the phase function $f(t)$, defined by the equation $f'(\gamma) = 0$, and $g(t)$ is an analytic function verifying that $g(0) = 0$. Integral (A1) can be evaluated asymptotically by using a generalized method of steepest descent (the classical method is not valid in the case of a saddle-point near zero).

On changing variables such that $t = t(z)$, we have

$$f(t;\gamma) = f(0;\gamma) + \mu\left\{\frac{z^2}{2} + \eta z\right\}$$

$$\mu = \mathrm{sgn}[f''(\gamma;\gamma)] \qquad \text{(A2)}$$

and the integral (A1) is written in the form

$$I(K) = \mathrm{e}^{\mathrm{i}Kf(0;\gamma)}\int_0^\infty g(t)\frac{\mathrm{d}t}{\mathrm{d}z}\mathrm{e}^{\mathrm{i}K\mu(\eta z + z^2/2)}\,\mathrm{d}z \qquad \text{(A3)}$$

where η is a function of the saddle-point γ. From (A2) we obtain

$$\eta = -(\mathrm{sgn}\,\gamma)\sqrt{2|f(\gamma;\gamma) - f(0;\gamma)|}. \qquad \text{(A4)}$$

In order to replace the exponent in (A3) by a real and negative quantity, the integration path is rotated by $\mu\pi/4$:

$$z = \zeta\exp(\mu\mathrm{i}\pi/4).$$

Thus, the integral (A3) is written as

$$I(K) = \exp[\mathrm{i}Kf(0;\gamma) + \mathrm{i}\mu\pi/4]\,J(K) \qquad \text{(A5)}$$

where

$$J(K) = \int_0^\infty G(\zeta)\mathrm{e}^{-K(b\zeta + \zeta^2/2)}\,\mathrm{d}\zeta \qquad \text{(A6)}$$

with $G(\zeta) = g(t)\,\mathrm{d}t/\mathrm{d}z$ and $b = \eta\mathrm{e}^{-\mathrm{i}\mu/4}$.

By expanding we get

$$G(\zeta) = \delta_0 + \zeta\delta_1 + \zeta(\zeta + b)G_1(\zeta)$$

where δ_0, δ_1 and G_1 will be determined. The integral (A6) is written in the form

$$J(K) = \delta_0\frac{W_0(\sqrt{K}b)}{K^{1/2}} - \delta_1\frac{W_0'(\sqrt{K}b)}{K} + J_1(K) \quad \text{(A7)}$$

where the function $W_0(s)$ is

$$W_0(s) = \int_0^{+\infty} \mathrm{e}^{-(st + t^2/2)}\,\mathrm{d}t$$

as defined from Weber's function $D_0(\mathrm{i}s)$ by the equation

$$W_0(s) = C\exp(s^2/4)\,D_0(\mathrm{i}s)$$

where $C = (2\pi)^{1/2}$ and $J_1(K)$ is defined by

$$J_1(K) = \int_0^{+\infty} \zeta(\zeta + b)G_1(\zeta)\exp\left[-K\left(\frac{\zeta^2}{2} + b\zeta\right)\right]\mathrm{d}\zeta. \qquad \text{(A8)}$$

After integration by parts, (A8) becomes

$$J_1(K) = \frac{1}{K}\int_0^{+\infty}[G_1(\zeta) + KG_1'(\zeta)] \times\exp\left[-K\left(\frac{\zeta^2}{2} + b\zeta\right)\right]\mathrm{d}\zeta. \qquad \text{(A9)}$$

We apply now the same procedure for the integral $J_1(K)$ as we did for the integral $J(K)$. By expanding

$$G_1(\zeta) + \zeta G_1'(\zeta) = \delta_2 + \zeta\delta_3 + \zeta(\zeta + b)G_2(\zeta)$$

in equation (A9), the integral (A7) can be written as

$$J(K) = \frac{W_0(\sqrt{K}b)}{K^{1/2}}\left(\delta_0 + \frac{\delta_2}{K}\right) - \frac{W_0'(\sqrt{K}b)}{K}\left(\delta_1 + \frac{\delta_3}{K}\right) + \frac{1}{K}J_2(K) \tag{A10}$$

with

$$J_2(K) = \int_0^{+\infty} \zeta(\zeta + b)G_2(\zeta)\exp\left[-K\left(\frac{\zeta^2}{2} + b\zeta\right)\right]\mathrm{d}\zeta. \tag{A11}$$

By continuing the same procedure, we obtain the final result in the form of an asymptotic series:

$$J(K) \approx \frac{W_0(\sqrt{K}b)}{K^{1/2}}\sum_{n=0}^{\infty}\frac{\delta_{2n}}{K^n} - \frac{W_0'(\sqrt{K}b)}{K}\sum_{n=0}^{\infty}\frac{\delta_{2n+1}}{K^n} \tag{A12}$$

where the constants δ_0 and δ_1 are given by

$$\delta_0 = G(0) = g(0)\left.\frac{\mathrm{d}t}{\mathrm{d}z}\right|_{z=0}$$

$$\delta_1 = \frac{G(0) - G(-b)}{b} = \eta^{-1} \times \mathrm{e}^{\mu i\pi/4}\left[g(0)\left.\frac{\mathrm{d}t}{\mathrm{d}z}\right|_{z=0} - g(\gamma)\left(\frac{\gamma}{-\eta}\right)\left.\frac{\mathrm{d}t}{\mathrm{d}z}\right|_{z=-\eta}\right]$$

and the function $W_0(\sqrt{K}b)$ is defined by

$$W_0(\sqrt{K}\eta\mathrm{e}^{-\mu i\pi/4}) = \exp\left[-\mu i\left(\frac{K\eta^2}{2} + \frac{\pi}{4}\right)\right]\int_{\eta\sqrt{K}}^{\infty}\exp\left(\mu i\frac{t^2}{2}\right)\mathrm{d}t.$$

By replacing (A12) in (A5), we finally obtain an analytical expression for the integral (A1):

$$I(K) \approx \exp(\mathrm{i}Kf(0) + \mathrm{i}\mu\pi/4) \times \left(\frac{\delta_0 + \eta\delta_1\mathrm{e}^{-\mu i\pi/4}}{K^{1/2}}W_0(\sqrt{K}\eta\mathrm{e}^{-\mu i\pi/4}) + \frac{\delta_1}{K}\right) \tag{A13}$$

with

$$\delta_0 - \eta\delta_1\mathrm{e}^{-\mu i\pi/4} = g(\gamma|f''(\gamma;\gamma)|^{-1/2}$$

$$\delta_1 = \frac{\mathrm{e}^{\mu i\pi/4}}{\eta}\left(g(0)\frac{\mu\eta}{f'(\gamma;\gamma)} - g(\gamma)|f''(\gamma;\gamma)|^{-1/2}\right).$$

In the case of a saddle-point equal to zero, expression (A13) can be simplified to

$$I(K) \approx \left[g(0)\left(\frac{\pi}{2K|f''(0)|}\right)^{1/2}\mathrm{e}^{-\mu i\pi/4} + \frac{g'(0)\mathrm{e}^{-\mu i\pi/4}}{K|f''(0)|}\right] \times \mathrm{e}^{\mathrm{i}Kf(0)}. \tag{A14}$$

References

[1] O'Neil H T 1949 Theory of focusing radiators *J. Acoust. Soc. Am.* **21** 510–26

[2] Cavanagh E and Cook B D 1981 Lens in the nearfield of a circular transducers: Gaussian Laguerre formulation *J. Acoust. Soc. Am.* **69** 345–51

[3] Cavanagh E and Cook B D 1980 Gaussian–Laguerre description of ultrasonic fields–numerical example: circular piston *J. Acoust. Soc. Am.* **67** 1136–44

[4] Cook B D and Arnoult W J 1976 Gaussian–Laguerre/Hermite formulation for the nearfield of an ultrasonic transducer *J. Acoust. Soc. Am.* **59** 9–11

[5] Lucas B G and Muir T G 1982 The field of a focusing source *J. Acoust. Soc. Am.* **72** 1289–96

[6] Filipczynski L and Etienne J 1973 Theoretical study and experiments of spherical focusing transducers with Gaussian surface velocity distribution *Acoustica* **28** 121–8

[7] Bertoni H L, Hsue C W and Tamir T 1985 Non-specular reflection of convergent beams from liquid–solid interface *Trait. Sign.* **2** 201–5

[8] Pott J and Harris J G 1984 Scattering of an acoustic Gaussian beam from a liquid–solid interface *J. Acoust. Soc. Am.* **76** 1829–38

[9] Pott J and Harris J G 1985 Further studies jof the scattering of an acoustic Gaussian beam from a liquid–solid interface *J. Acoust. Soc. Am.* **78** 1072–80

[10] Rousseau M and Gatignol P 1986 Analyse asymptotique des faisceaux ultrasonores et problèmes d'interface *J. Theor. Appl. Mech.* Special issue 273–93

[11] Rousseau M and Gatignol P 1988 Etude asymptotique d'un faisceau Gaussien focalisé *J. Acoust.* **1** 95–9

[12] Matikas T E, Rousseau M and Gatignol P 1993 Theoretical analysis for the reflection of a focused ultrasonic beam from a fluid–solid interface *J. Acoust. Soc. Am.* **93** 1407–16

[13] Matikas T E 1991 Asymptotic and experimental analysis of the nonspecular reflection of focusing ultrasonic beams from fluid–solid interface *Acoustical Imaging* vol 19 (Bochum: Plenum) pp 133–9

[14] Bleinstein N 1966 Uniform asymptotic expansion of integrals with stationary point near algebraic singularity *Comm. Pure Appl. Math.* **XIX** 353–70